

# ON EXCESS OF THE ODIIOUS PRIMES

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ABSTRACT. We give a more strong heuristic justification of our conjecture on the excess of the odious primes.

## 1. INTRODUCTION

This note is a continuation of the author's paper [4]. Until now the author did not know about the Moser's digit conjecture and its solutions in [3] and [1]. In fact, we gave in [4] a new combinatorial proof of this conjecture (see Theorem 3[4]) and proved also an addition to the Moser-Newman theorem for the excess of the evil nonnegative odd integers less than  $n$  and divisible by 3 (see Theorem 4 and 5 in [4]).

The aim of the present note - to give a more strong heuristic justification ("almost strong proof") of our Conjectures 1 and 2 [4].

Recently in their excellent research [2], Mauduit and Rivat solved the Gelfond digit problem for primes. In particular, they proved that

$$(1) \quad \lim_{n \rightarrow \infty} \frac{\pi^0(n)}{\pi(n)} = \lim_{n \rightarrow \infty} \frac{\pi^e(n)}{\pi(n)} = \frac{1}{2}.$$

Moreover, if  $\pi_{3,i}^0(n)(\pi_{3,i}^e(n))$ ,  $i = 1, 2$ , is the number of the odious (evil) primes  $p \equiv i \pmod{3}$  not exceeding  $n$  and

$$\pi_{3,i}(n) = \pi_{3,i}^0(n) + \pi_{3,i}^e(n), \quad i = 1, 2,$$

they also proved that

$$(2) \quad \lim_{n \rightarrow \infty} \frac{\pi_{3,i}^0(n)}{\pi_{3,i}(n)} = \lim_{n \rightarrow \infty} \frac{\pi_{3,i}^e(n)}{\pi_{3,i}(n)} = \frac{1}{2}$$

These results mean that the events " $n$  is a prime" and " $n$  is an odious integer" are asymptotic independent for large  $n$ .

In turn, this means that the odious-evil asymptotic behavior of the primes of the form  $3k + 1(3k + 2)$  is proportionally similar to the odious-evil asymptotic behavior of all odd integers of the form  $3k + 1(3k + 2)$ .

## 2. PROOF OF CONJECTURES

Let  $\mu^0(n)(\mu^e(n))$  be the number of *odd* odious (evil) integers less than  $n$ .

**Lemma 1.**  $|\mu^0(n) - \mu^e(n)| \leq 1$ .

*Proof.* The lemma follows from the identity

$$(3) \quad \mu^0(4m+1) = \mu^e(4m+1), \quad m \in \mathbb{N},$$

which is proved by induction.

Notice that (3) is valid for  $m = 1$ . Assuming that it is valid for  $4m+1$  we prove (3) for  $4(m+1)+1$ . Indeed, let  $m$  have  $k$  ones in the binary expansion. Then taking into account that for odd  $k$  the number  $4m+1$  is evil and for even  $k$  the number  $4m+1$  is odious, and using the induction conjecture we have

$$\mu^0(4m+3) - \mu^e(4m+3) = (-1)^k.$$

Furthermore,  $4m+3$  is evil if  $k$  is even and is odious if  $k$  is odd. Therefore,

$$\mu^0(4m+5) - \mu^e(4m+5) = (-1)^k + (-1)^{k+1} = 0. \blacksquare$$

Let  $\Delta_{3,i}(n)(\Delta_{3,i}^{odd}(n))$ ,  $i = 0, 1, 2$  be the excess of the (*odd*) odious integers  $m \in [0, n)$  such that  $m \equiv i \pmod{3}$ .

In particular, according to the notations of [4]

$$(4) \quad \Delta_{3,0}^{odd}(n) = -\Delta_3^{odd}([0, n)) < 0, \quad \Delta_{3,0}(n) = -\Delta_3([0, n)) < 0.$$

Let, furthermore,  $\Delta_{3,i}^{primes}$ ,  $i = 1, 2$  be the excess of odious odd primes  $p \in [0, n)$  such that  $p \equiv i \pmod{3}$ . Then by (1),(2) taking into account the independence of the above mentioned events, in the case of  $|\Delta_{3,i}^{odd}(n)| \gg \ln n$  we have

$$(5) \quad \Delta_{3,i}^{primes} \sim \frac{3\Delta_{3,i}^{odd}(n)}{\ln n}, \quad i = 1, 2.$$

So, for  $i = 1$  and for even  $n$  we have

$$(6) \quad \Delta_{3,1}^{odd}(n) = -\Delta_{3,0}\left(\frac{n}{2}\right) = \Delta_3\left([0, \frac{n}{2})\right).$$

Newman showed [3], that for all  $n \in \mathbb{N}$

$$(7) \quad (0.05 \cdot 3^\alpha)n^\alpha \leq \Delta_3([0, n)) \leq (5 \cdot 3^\alpha)n^\alpha \quad \text{with } \alpha = \frac{\ln 3}{\ln 4}.$$

Therefore, by (6)

$$(8) \quad \Delta_{3,1}^{odd}(n) \geq 0.05(1.5)^\alpha n^\alpha \gg \ln n.$$

In the case of  $i = 2$  the absolute value of the excess  $\Delta_{3,2}^{odd}(n)$  is small for some  $n$ . Indeed, by Lemma 1

$$(9) \quad \Delta_{3,0}^{odd} + \Delta_{3,1}^{odd} + \Delta_{3,2}^{odd} = \delta_n,$$

where  $|\delta_n| \leq 1$ .

Thus by (5) and (6)

$$(10) \quad \Delta_{3,2}^{odd} = \delta_n - \Delta_{3,0}^{odd} - \Delta_{3,1}^{odd} = \Delta_3^{odd}([0, n)) - \Delta_3\left(\left[0, \frac{n}{2}\right)\right) + \delta_n.$$

With help of (10) and the exact formulas for  $\Delta_n^{odd}(n)$ ,  $\Delta_3(n)$  [4] we obtain in particular that

$$(11) \quad \Delta_{3,2}^{odd}([0, 2^{2n-1})) = -3^{n-2}, \quad \Delta_{3,2}^{odd}([0, 2^{2n})) = 0.$$

Nevertheless, it is sufficient for us to understand (5) for small  $|\Delta_{3,2}^{odd}(n)|$  by the following way: if

$$(12) \quad |\Delta_{3,2}^{odd}(n)| \leq \sqrt{n} \text{ then } |\Delta_{3,2}^{primes}(n)| = O\left(\frac{\sqrt{n}}{\ln n}\right).$$

Now if  $|\Delta_{3,2}^{odd}| > \sqrt{n}$  by (5), (6) and (10) we have

$$(13) \quad \pi^0(n) - \pi^e(n) = \Delta_{3,1}^{primes}(n) + \Delta_{3,2}^{primes}(n) \sim \frac{3\Delta_3^{odd}([0, n))}{\ln n}.$$

Note that, according to [4]

$$(14) \quad \lim_{n \rightarrow \infty} \frac{\ln \Delta_3^{odd}([0, n))}{\ln n} = n^\alpha.$$

If  $|\Delta_{3,2}^{odd}| \leq \sqrt{n}$  then by (5), (6) and (12) we have

$$(15) \quad \pi^0(n) - \pi^e(n) = \frac{3\Delta_{3,1}^{odd}(n)}{\ln n}(1 + o(1)) + O(\sqrt{n}) \sim \frac{3\Delta_3([0, \frac{n}{2}])}{\ln n}.$$

Now by (8), (13)-(15) we find as the final result that

$$\ln(\pi^0(n) - \pi^e(n)) = \frac{\ln 3}{\ln 4} \ln n + o(\ln n)$$

and our Conjecture 2 follows. ■

Note that from Conjecture 2 evidently follows the statement of Conjecture 1 but only for sufficiently large  $n \geq n_0$ . Unfortunately, until now we are not able to estimate  $n_0$ .

Note that by the way we obtain the limits

$$\lim_{n \rightarrow \infty} \frac{\ln(\pi_{3,1}^o(n) - \pi_{3,1}^e(n))}{\ln n} = \frac{\ln 3}{\ln 4};$$

for  $n_k = 2^{2k-1}$ ,  $k \in \mathbb{N}$ ,

$$\lim_{k \rightarrow \infty} \frac{\ln(\pi_{3,2}^e(n_k) - \pi_{3,2}^o(n_k))}{\ln n_k} = \frac{\ln 3}{\ln 4}.$$

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